TIME–FREQUENCY PERSPECTIVES: THE "CHIRPLET" TRANSFORM

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Abstract

We have developed a new expansion[1] we call the "chirplet transform". It has been successfully applied to a wide variety of signal processing applications including radar[1] and image processing.

There has been a recent debate as to the relative merits of an affine–in–time (wavelet) transform and the classical Short Time Fourier Transform (STFT), for the analysis of non-stationary phenomena. Chirplet filters embody both the wavelet and STFT as special cases by decoupling the filter bandwidths and center frequencies.

Chirplets, by their embodiment of affine geometry in the TF plane, may also include shears in time and frequency (chirps), and even time–bandwidth product variation (noise bursts) if desired. The most general chirplets may be derived from one or more *basic* ("mother") chirplets by the transformations of perspective geometry in the Time–Frequency (TF) plane.

1 INTRODUCTION

The well–known wavelet transform was originally derived through one dimensional affine transformations in the physical (eg. time) domain. Our proposed chirplet bases, however, were first derived through affine transforms in the TF plane, and later through the application of projective geometry to the TF plane. One way to visualize this process is to imagine that you have some basic ("mother") chirplet. You compute its Time–Frequency distribution (eg. Wigner distribution, spectrogram, or the like). You then photograph the display of this distribution at any oblique angle. For each photograph of the distribution, you hypothetically derive an inverse of that TF distribution to arrive at a new time–domain function. In its most general form, the family of chirplets is defined by the eight parameters of *twice–applied perspective geometry*. An alternative visualization is to imagine grabbing the four corners of the mother chirplet's TF distribution and moving them wherever you like. Now the family of chirplets are the functions which have these new TF distributions.

Of course, computation of the full eight parameter chirplets themselves is not trivial, and a transform derived as an array with eight incices is horrendous both to compute and to store digitally, with any reasonable grid density. Nevertheless, this new mode of thought gives us another perspective (no pun intended) on TF theory by providing a unified framework in which to view most other TF methods which are embodied as lower dimensional manifolds in the new "chirplet" space. For example, both the wavelet transform, and the Short Time Fourier Transform (STFT) are planar chirplet slices. Many adaptive methods may be expressed as two–dimensional chirplet manifolds. Baraniuk *et. al.* [2][3] provide an excellent treatment of a number of adaptive TF methods.

[1] Papers in PostScript, including figures, available by e-mail.

2 SPECIAL CASES

We have identified various special cases of the chirplet, along with their number of free parameters, and successful practical applications. The cases are arranged in a hierarchical order; each one is a special case of all of the ones above it.

- Co-linear; approximate with 8 parameters; one (T,F) coordinate pair for each corner of the "control box"; (Bi–linear interpolation may be used if a rough approximation is sufficient.) Used for modeling processes such as guitar strings which have a broad (noiselike) spectrum when initially plucked, but become more tonelike as the process evolves.
- Perspective; 7 parameters; We have extended our idea to higher dimensions for image processing applications[4]. Consider an oblique image of a tiled wall. In the picture, the "periodic" pattern

gets smaller and smaller toward the end of the picture where the tiles were further from the camera. Decomposing onto a basis of "plane chirps" performs better than 2–D Fourier decomposition (plane waves) because plane chirps may vary in density (spatial variation in spatial frequency) to match arbitrary perspective.

- Affine: 6 parameters, of the form $\mathbf{y} = \mathbf{A}\mathbf{x}+\mathbf{b}$ where $\mathbf{A} \in \Re^{2\times 2}$ and $\mathbf{b} \in \Re^2$; Successfully applied to marine radar. May generate noiselike bursts, for modeling Doppler returns from sea clutter, or other nonrigid bodies.
- Symplectic (constant time–bandwidth product: $det|\mathbf{A}| = 1$); 5 parameters; Tone–like bases model rigid bodies moving at constant speeds. Pure chirp bases model Doppler from rigid bodies undergoing acceleration. Successfully used for automobile traffic radar, as well as marine radar targets.
- T and F Shear Invariant; 4 parameters. Mother chirplets for which no distinction between shear–in–time and shear–in–frequency need be made. Examples include the chirped Gabor bases, if we consider magnitude TF distributions only. Successfully applied to radar.

3 TF–AFFINE CHIRPLETS

Here we will only examine the TF–affine chirplet.

Using the six 2–D affine transformations in the TF plane leads to a transform which is stored in an array having six indices.[2] (Recall that the wavelet bases have only two parameters, dilation and translation, since their affinity is only in the physical domain.)

This basis consists of all the members of a particular time domain signal, which are affine transformations of each other when viewed in TF space. The TF distributions of these bases could all be thought of as being from the mother chirplet's TF distribution, viewed obliquely through a telescope (rectangles become parallelograms). In other words, no perspective is involved in the TF–affine transformations.

Philosophically there are two ways to think of the chirplet:

1. Using the "piece–of–a–chirp" framework. This philosophy is exemplified in Figure 1
2. Thinking in terms of affine transformations in TF space, which consist of dilations and "chirpings" (in

[2] The large number of degrees of freedom was dealt with either by examining lower dimension manifolds, or by using an adaptive algorithm[5].

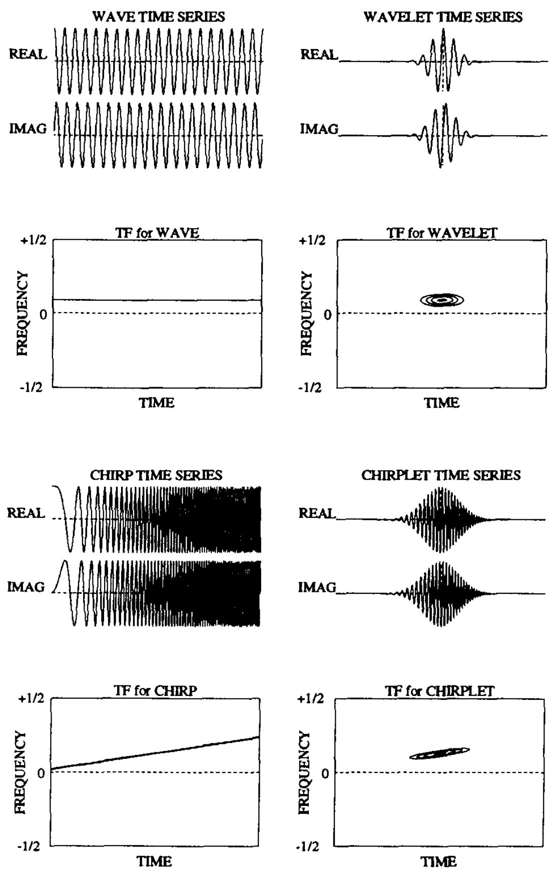

Figure 1: Relationship between wave, wave**let**, chirp and "chirp**let**", in terms of time series and magnitude Time–Frequency (TF) distributions. We have extended the one dimensional affinity of the wave**let** to two dimensions, by adding up–down translation and shear in TF space. These extra affine transformations are achieved by multiplication, in the time domain, of the wavelet by a chirp, which performs a *shear* and a translation along the **Frequency** axis. This simplified Gabor chirplet, with only 4 parameters, assumes we are looking at magnitude–only TF distributions.

both time and frequency). We note that translations (modulations and delays) are just special cases of "chirpings" (in time and frequency), where the chirp rate is zero. This second philosophy is illustrated in Figure 2.

4 THE PROLET CHIRPLET

We illustrate our TF affine concept by a simple example, using a function which "attempts to be" rectangular in TF space, the Discrete Prolate Spheroidal Sequence (DPSS). These functions are of special interest in the signal processing community (for a full description, the reader is referred to Landau, Pollack, Slepian[6] [7]) and are commonly referred to as *prolates* or *Slepians*[3]. When we apply our TF affine transformations to the prolate, we obtain a specific class of chirplets which we refer to as prolate chirplets [4].

We define six operators which act on a mother chirplet, $g(t)$, to produce the other members of the family. The symbols depict the change in shape within the TF plane, where the arrows indicate the sign convention chosen, and do not necessarily indicate the actual direction of operation for all the members of the family.

1. $\boxed{\substack{\rightarrow\\ \rightarrow}}(t_0)g(t) = g(t - t_0)$ is a translation in time, by t_0. A shift right in TF space is the same as a shift right in time. An interesting, although very inefficient alternative means of performing this translation is by a Fourier transformation, followed by (de)modulation, followed by an inverse Fourier transformation. This seemingly obscure conceptualization helps in making the mental jump to the shear–in–time operation.

2. $\boxed{\uparrow\uparrow}(f_0)g(t) = g(t)e^{j2\pi f_0}$ is a translation in frequency (modulation).

3. $\boxed{\substack{\leftrightarrow\\ \leftrightarrow}}(a)g(t) = h(t/a)$ is a dilation in time, by scale factor a, with a new function h which has time bandwidth product also increased by a. (Mathematical description depends on g.)

4. $\boxed{\updownarrow\updownarrow}(a)g(t) = h(at)$ is a dilation in frequency, by scale factor a, with increase in time bandwidth product. (Again, mathematical description depends on g.)

[3] We have applied the TF–affine operators to a number of different signals. Here the Slepian is chosen simply because it is the most illustrative of the concept, not necessarily because it gives the best performance.

[4] We have also successfully implemented a new variant of Thomson's method of spectral estimation using a family of these prolate chirplets as multiple data windows.

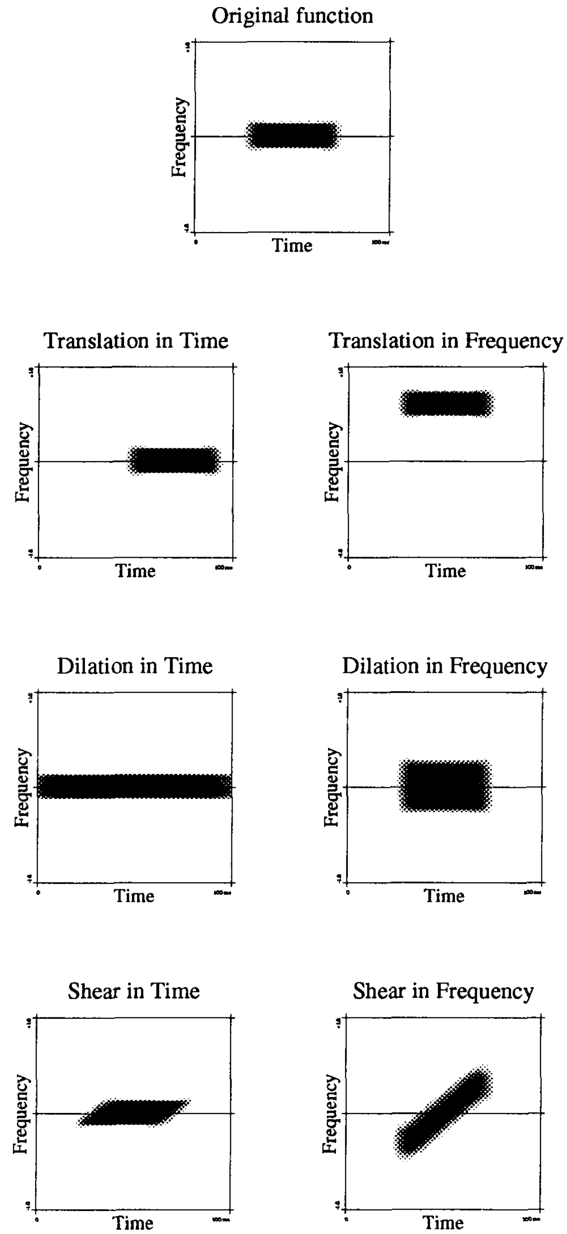

Figure 2: Actual spectrograms of a particular mother chirplet, after being acted on by each of the TF–affine transformations. Here we have used a family of Discrete Prolate Spheroidal Sequences (DPSS), and arrived at our 6 parameter prolate chirplet family. (Note that the time bandwidth product is free to vary; although it has a lower bound. We may also fix the time bandwidth product, leaving 5 independent parameters, and simplifying the mathematical description.)

5. $\boxed{\rightleftarrows}(t_b, t_e)g(t) = \mathcal{F}^{-1}C(-t_b, -t_e)\mathcal{F}g(t)$, where $\mathcal{F}$ designates the Fourier operator, and $C(x_b, x_e)g(x) = exp(j2\pi(\frac{x_e - x_b}{2}x + \frac{x_e + x_b}{2}x)g(x)$ is the "chirping" operation, which may be defined in terms of beginning and ending time (or frequency). The chirping operator is simply an extension of (de)modulation to two time or frequency variables, rather than just one. Thus a shear in time is simply a chirping in the frequency domain.

6. $\boxed{\downarrow\uparrow}(f_b, f_e)g(t) = C(f_b, f_e)g(t)$ is a shear in frequency which is just an extension of modulation to both a beginning frequency and an ending frequency, rather than just a center frequency as in the usual definition of modulation.

The effect of each of these six operators is illustrated as a TF density plot in Figure 2. Actual spectrograms were computed using Thomson's method of spectral estimation, but any TF method gives roughly the same overall shape, with a similar 2–D affine characteristic.

Note also that if we simply stretch out one of the chirplets in time, we also compact it in frequency. In some applications, it may not be necessary to independently dilate in time and frequency, so that that operators 3 and 4 may be combined into one operator as follows: $\boxed{\overset{\leftrightarrow}{\leftrightarrow}}(a)g(t) = g(t/a)$, giving a 5 parameter space, where the mathematical description of all the members of the chirplet family is of the same form. This symplectic case corresponds to the constant time–bandwidth product previously mentioned, where the term "chirplet" is perhaps most appropriate (ie. *pure* chirps).

Many chirplet families fall into this category. One of the notable exceptions, however, is a family of DPSS which may collectivly act to define a mother chirplet with possibly increased time–bandwidth product.

5 "WARBLETS"

Tests on actual radar data, pertaining to ocean surveillance, show that the radar return from small ice fragments rises and falls in frequency. (From a surfer's perspective ocean waves rise and fall periodically, so it stands to reason that the Doppler tone (velocity) one gets from a floating object also rises and falls periodically in pitch.)

In order to match this physical phenomenon we selected a particular "mother chirplet", to which we applied the first three of the our TF–affine operators, along with the constant *time–bandwidth product* constraint. Since this particular choice of chirplet has a profound significance, we have given it a special name, the "warblet". Warblets are chirplets where the *mother chirplet* is a single tone FM signal (like the sound produced by either a police siren or the bird known as a *warbler*), as given by: $\psi = Ae^{j(\frac{\beta sin(2\pi f_m t + p)}{f_m}) + j2\pi f_c t}$

A particular manifold in warblet space, the modulation–index versus modulation–frequency plane, has been found to be very useful in analyzing actual marine radar data, making use of the nearly cyclo–stationary TF pattern of Doppler returns from floating objects being influenced by ocean waves.

6 CONCLUSION

We have created a new paradigm in TF analysis which allows for a general framework in which to discuss and compare various methods. Although the chirplet transform is seldom actually used in its entire form, it lends itself to various subspaces, which have proven themselves in a number of practical applications. Furthermore, we have laid the foundation for further development of this general theory.

References

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